

THE PENROSE TRANSFORM IN QUATERNIONIC GEOMETRY

RADU PANTILIE

ABSTRACT. We study the Penrose transform for the ‘quaternionic objects’ whose twistor spaces are complex manifolds endowed with locally complete families of embedded Riemann spheres with positive normal bundles.

INTRODUCTION

The obvious aim, when trying to solve an equation, is to find all the solutions, explicitly. However, one is usually satisfied with a subclass of solutions and a procedure by which all the solutions can be obtained from those in the subclass.

The (arche)typical example is given by the Laplacian on $\mathbb{R}^2(=\mathbb{C})$, where any complex-valued harmonic function is, locally, the sum of a holomorphic and an anti-holomorphic function.

On \mathbb{R}^3 , the situation was settled, essentially, in [19]: for any complex-valued harmonic function f , locally defined on \mathbb{R}^3 , there exists a family of maps $\varphi_t : \mathbb{R}^3 \rightarrow \mathbb{C}$, with $t \in I$, and $I \subseteq \mathbb{R}$ a closed interval, such that $\varphi_t = \psi_t \circ \pi_t$, with $\pi_t : \mathbb{R}^3 \rightarrow \mathbb{C}$ orthogonal projections, ψ_t locally defined holomorphic functions on \mathbb{C} , and

$$(0.1) \quad f = \int_I \varphi_t dt .$$

On \mathbb{R}^4 , with the Lorentzian signature, this was done by R. Penrose (see [13] and the references therein). Up to a complexification (in Twistor Theory, the objects are usually real-analytic) we may assume Riemannian signature.

The procedure by which all of the harmonic functions on \mathbb{R}^4 can be found is given, again, by (0.1), whilst the particular class of solutions is formed of all the functions φ for which there exists a (positive) Kähler structure on \mathbb{R}^4 , with respect to which φ is holomorphic.

The first example, above, admits the obvious higher dimensional generalization through the classical notion of pluriharmonic function, of complex analysis. Furthermore, the unicity is easy to deal with.

On the other hand, neither the unicity nor the adequate level of generality can be easily described for the other two examples. Nevertheless, the former problem can be understood through the classical fact that the integral representations above can be obtained, through the Čech cohomology, from elements of $H^1(Z, \mathcal{L}^2)$, where $Z = \mathcal{O}(2)$, for \mathbb{R}^3 , $Z = \mathcal{O}(1) \oplus \mathcal{O}(1)$, for \mathbb{R}^4 , and $\mathcal{L} = \pi^*(\mathcal{O}(-1))$, with $\mathcal{O}(k)$ the holomorphic

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line bundle of Chern number k over the Riemann sphere, and π the bundle projection of Z . That is, *the Penrose transform* establishes a correspondence between certain cohomology classes over the twistor space Z of M and the ‘pluriharmonic functions’ on M .

In this paper, we study the Penrose transform in the case when, up to conjugations, the twistor space is a complex manifold endowed with a locally complete family of embedded Riemann spheres with positive normal bundles. On the differential geometric side, these correspond to what we call ‘quaternionic objects’ (see Section 2, below). This way, we retrieve, for example, the already known cases of (classical) quaternionic manifolds [3], and, in particular, the anti-self-dual manifolds [6].

The fact that the parameter spaces of families of embedded Riemann spheres are, canonically, ‘quaternionic’ has its roots in the fact that the quaternions arise from the Riemann sphere endowed with the antipodal map. This is explained in Section 1 which we believe will help the reader to understand the quaternionic objects and the corresponding Penrose transform.

The relevant notion of ‘quaternionic pluriharmonicity’ is constructed in Section 3. This is done in two steps, by using the fact that any suitable holomorphic line bundle over the twistor space of M corresponds to a ‘hypercomplex object’ which is the total space of a principal bundle over M (see Definition 3.1, and the comments before it).

Finally, the quaternionic Penrose transform is discussed in Section 4. We believe that our approach is as explicit as possible, based on a cohomology result of [4], and in the particular case of anti-self-dual manifolds it, essentially, reduces to the approach of [20] (see, also, [15]).

1. WHERE DO THE QUATERNIONS COME FROM?

The idea is that *the quaternions arise from the Riemann sphere* (endowed with the antipodal map), as we shall now explain.

A *Riemann sphere* is a compact Riemann surface Y , with zero first Betti number, endowed with an involutive antiholomorphic diffeomorphism σ without fixed points. From the exact sequence of cohomology groups associated to the exponential sequence $\{0\} \rightarrow \mathbb{Z} \rightarrow \mathbb{C} \rightarrow \mathbb{C}^* \rightarrow \{1\}$ we deduce that $H^1(Y, \mathbb{C}^*) = \mathbb{Z}$. In particular, each holomorphic line bundle over Y is determined, up to isomorphisms, by its Chern number. Furthermore, each element of $H^1(Y, \mathbb{C}^*)$ is determined by a divisor. Thus, on denoting by \bar{Y} the Riemann surface with the same underlying conformal structure as Y but with the opposite orientation, σ induces an isomorphism $H^1(Y, \mathbb{C}^*) = H^1(\bar{Y}, \mathbb{C}^*)$ which preserves the Chern numbers.

On the other hand, if \mathcal{L} is a holomorphic line bundle over Y then its conjugate $\bar{\mathcal{L}}$ is a holomorphic line bundle over \bar{Y} . Consequently, \mathcal{L} is isomorphic to $\sigma^*(\bar{\mathcal{L}})$. Equivalently, there exists an antiholomorphic diffeomorphism $\tau : \mathcal{L} \rightarrow \mathcal{L}$, covering σ , which is complex-conjugate linear on each fibre. Moreover, any two such morphisms differ by the

multiplication with a nonzero complex number. Furthermore, as τ^2 is an isomorphism of \mathcal{L} (covering the identity map of Y) we have $\tau^2 = \lambda \text{Id}_{\mathcal{L}}$, for some nonzero complex number λ . But τ^2 is, also, an isomorphism of $\overline{\mathcal{L}}$ which implies that λ is real. Therefore, by suitably multiplying, if necessary, τ with a complex number, we may assume $\tau^2 = \pm \text{Id}_{\mathcal{L}}$. Note that, τ is unique, with this property, up to a factor of modulus 1.

Proposition 1.1 (cf. [14]). $\tau^2 = (-1)^d \text{Id}_{\mathcal{L}}$, where d is the Chern number of \mathcal{L} .

Proof. Suppose that \mathcal{L} has Chern number 1. By using the Kodaira vanishing and the Riemann–Roch theorems, we obtain that $H = H^0(Y, \mathcal{L})$ has (complex) dimension two. Consequently, the elements of H which vanish at a point of Y form a one-dimensional subspace of H . Therefore we may identify $Y = PH$ and, together with the fact that σ has no fixed points, we deduce $\tau^2 = -\text{Id}_{\mathcal{L}}$.

Now, the proof follows quickly from the fact that any holomorphic line bundle over Y , of Chern number d , is of the form \mathcal{L}^d , where \mathcal{L} has Chern number 1. \square

Remark 1.2. Let \mathcal{L} be a holomorphic line bundle, of Chern number -1 , over a Riemann sphere Y , and let $H = H^0(Y, \mathcal{L}^*)$. The proof of Proposition 1.1 shows that, under the identification $Y = PH(= P(H^*))$, \mathcal{L} becomes the tautological line bundle, canonically embedded into the trivial bundle $Y \times H^*$.

We can now make the following:

Definition 1.3. Let Y be the Riemann sphere and let $H = H^0(Y, \mathcal{L})$, where \mathcal{L} is a holomorphic line bundle over Y of Chern number 1.

The *algebra of quaternions* is the unital associative subalgebra $\mathbb{H} \subseteq \text{End}H$ formed of the elements which commute with the complex-conjugate isomorphism induced by τ . (Briefly, \mathbb{H} is formed of the ‘real’ elements of $\text{End}H$.)

The real multiples of Id_H are the *real quaternions*, whilst the trace-free elements of $\mathbb{H} \subseteq \text{End}H$ are the *imaginary quaternions*. Also, the (complex) determinant on $\text{End}H$ restricts to give the Euclidean structure of \mathbb{H} . Then the imaginary quaternions of modulus 1 are linear complex isomorphisms of H of square $-\text{Id}_H$. Thus, we obtain the identification $S^2 = PH(= Y)$ through which any imaginary quaternion of modulus 1 is identified with its eigenspace corresponding to $-i$. Then, under this identification, σ is the antipodal map.

Note that, \mathbb{H} does not depend of the choice of τ , but only of Y , σ , and \mathcal{L} . Moreover, if \mathcal{L}' is another holomorphic line bundle, of Chern number 1, and $H' = H^0(Y, \mathcal{L}')$ then the two embeddings of \mathbb{H} into $\text{End}H$ and $\text{End}H'$ differ by a composition with an element of $\text{SO}(3)$ acting trivially on \mathbb{R} and canonically on $\mathbb{R}^3(= \text{Im}\mathbb{H})$. In particular, the automorphism group of \mathbb{H} is $\text{SO}(3)$.

2. THE MAIN OBJECTS OF QUATERNIONIC GEOMETRY

In this section, we define the class of manifolds for which we want to describe the Penrose transform. Up to integrability, the corresponding geometric structure is given by

the almost co-CR quaternionic structures of [8, 9], whilst to formulate the integrability we need the generalized connections of [11]. We start by recalling the latter.

2.1. Principal ρ -connections. Let (P, M, G) be a principal bundle and let F be a vector bundle over M , endowed with a morphism of vector bundles $\rho : F \rightarrow TM$. A *principal ρ -connection* on P is a G -invariant morphism of vector bundles $C : \pi^*F \rightarrow TP$ such that $\widetilde{d\pi} \circ C = \pi^*\rho$, where $\pi : P \rightarrow M$ is the projection and $\widetilde{d\pi} : TP \rightarrow \pi^*(TM)$ is the morphism of vector bundles induced by $d\pi$.

The notion of associated connection has a straight generalization to this setting, as we shall now explain. Let S be a manifold on which G is acting to the left. Denote by $Y = P \times_G S$ the associated bundle, by π_Y its projection, and by $\mu : TP \times S \rightarrow TY$ the obvious G -invariant morphism of vector bundles, covering the projection $P \times S \rightarrow Y$.

On identifying, as usual, $\pi_Y^*(F)$ with a submanifold of $Y \times F$, we may define the *associated ρ -connection* $c : \pi_Y^*(F) \rightarrow TY$, as follows: $c([u, s], f) = \mu(C(u, f), s)$, for any $(u, f) \in \pi^*F$ and $s \in S$, where $[u, s] \in Y$ is the equivalence class determined by $(u, s) \in P \times S$. Note that, if we compose c with the morphism of vector bundles from TY onto $\pi_Y^*(TM)$, induced by $d\pi_Y$, we obtain $\pi_Y^*(\rho)$. Indeed, as $d\pi_Y \circ \mu$ is the composition of the projection from $TP \times S$ onto TP followed by $d\pi$, we have $(d\pi_Y \circ c)([u, s], f) = (d\pi \circ C)(u, f) = \rho(f)$, for any $(u, f) \in \pi^*F$ and $s \in S$.

Remark 2.1. Let (P, M, G) be a principal bundle endowed with a principal ρ -connection $C : \pi^*F \rightarrow TP$, where F is a vector bundle over M , and $\rho : F \rightarrow TM$ a morphism of vector bundles.

Suppose that $E \subseteq F$ is a vector subbundle mapped isomorphically by ρ onto a distribution on M . Then, by restriction, C induces a partial principal connection on P over $\rho(E)$. This, obviously, works, also, if $E \subseteq F^{\mathbb{C}}$ is a complex vector subbundle mapped isomorphically by (the complexification of) ρ onto a complex distribution on M .

If S is a vector space on which G acts by linear isomorphisms then any (associated) ρ -connection on the associated vector bundle $Y = P \times_G S$ corresponds to a covariant derivation $\nabla : \Gamma(Y) \rightarrow \Gamma(\text{Hom}(F, Y))$ which is a linear map satisfying $\nabla(fs) = \rho^*(df) \otimes s + f(\nabla s)$, for any function f on M and any section s of Y . If $Y = F$ then we can define the *torsion* of ∇ which is the section T of $TM \otimes \Lambda^2 F^*$ characterised by $T(s_1, s_2) = \rho(\nabla_{s_1}s_2 - \nabla_{s_2}s_1) - [\rho(s_1), \rho(s_2)]$, for any sections s_1, s_2 of F .

2.2. Quaternionic objects. A *linear quaternionic structure* on a (real) vector space F is an equivalence class of morphisms of associative algebras from \mathbb{H} to $\text{End}F$, where two such morphisms σ_1 and σ_2 are equivalent if $\sigma_2 = \sigma_1 \circ a$, for some $a \in \text{SO}(3)$. If $\sigma : \mathbb{H} \rightarrow \text{End}F$ is a morphism of associative algebras then the induced linear quaternionic structure on F is determined by $\sigma(S^2)$, whose elements are the *admissible linear complex structures* on F . Thus, a linear quaternionic structure is given by a family of linear complex structures, parameterized by the Riemann sphere.

A *linear co-CR quaternionic structure* on a vector space U is a pair (F, ρ) , where F is a quaternionic vector space, and $\rho : F \rightarrow U$ is a surjective linear map such that $(\ker \rho) \cap J(\ker \rho) = \{0\}$, for any admissible linear complex structure J on F ; consequently, $C = \rho(\ker(J + i))$ is a linear co-CR structure on U (that is, $C + \overline{C} = U^{\mathbb{C}}$). Thus, a linear co-CR quaternionic structure is given by a family of linear co-CR structures, parameterized by the Riemann sphere.

These notions extend in the obvious way to vector bundles, thus, giving the notion of almost co-CR quaternionic structure on a manifold. Note that, if F is a quaternionic vector bundle then the corresponding space Y of admissible linear complex structures on (the fibres of) F is a sphere bundle, with structural group $\mathrm{SO}(3)$.

Now, let (F, ρ) be an almost co-CR quaternionic structure on M and suppose that F is endowed with a ρ -connection (compatible with its structural group). Denote by Y the bundle of admissible linear complex structures on F and let $c : \pi^*F \rightarrow TY$ be the associated ρ -connection on it, where $\pi : Y \rightarrow M$ is the projection.

Let $\mathcal{B} \subseteq T^{\mathbb{C}}Y$ be the complex distribution given by $\mathcal{B}_J = c_J(\ker(J + i))$, for any $J \in Y$. Then $\mathcal{C} = \mathcal{B} \oplus (\ker d\pi)^{0,1}$ is an almost co-CR structure on Y ; that is, $\mathcal{C} + \overline{\mathcal{C}} = T^{\mathbb{C}}Y$.

Definition 2.2. We say that (M, F, ρ, c) is a *quaternionic object* if \mathcal{C} is integrable (that is, its space of sections is closed under the usual bracket).

Note that, a quaternionic object is a (real) ρ -quaternionic manifold [11] with ρ surjective. In particular, if ρ is an isomorphism then we obtain the classical notion of quaternionic manifold.

To define the corresponding twistor spaces, suppose that (M, F, ρ, c) is a quaternionic object for which there exists a surjective submersion $\psi : Y \rightarrow Z$ such that:

- (1) $(\ker d\psi)^{\mathbb{C}} = \mathcal{C} \cap \overline{\mathcal{C}}$,
- (2) \mathcal{C} is projectable with respect to ψ (note that, this is a consequence of (1), if the fibres of ψ are connected),
- (3) ψ restricted to each fibre of π is injective.

Then Z endowed with $d\psi(\mathcal{C})$ is a complex manifold (with $T^{0,1}Z = d\psi(\overline{\mathcal{C}})$) which is the *twistor space* of (M, F, ρ, c) . Furthermore, $(\psi(Y_x))_{x \in M}$, where $Y_x = \pi^{-1}(x)$, is a smooth family of embedded Riemann spheres on Z , whose members are the *real twistor spheres*.

Example 2.3. Let (U, F, ρ) be a co-CR quaternionic vector space. Then $U \times F$ is a quaternionic vector bundle over U and $\mathrm{Id}_U \times \rho$ is a morphism of vector bundles from $U \times F$ onto $TU (= U \times U)$, giving an almost co-CR quaternionic structure on U .

Moreover, on endowing $U \times F$ with the obvious (classical) flat connection, we obtain a quaternionic object whose twistor space is the holomorphic vector bundle \mathcal{U} , over the Riemann sphere, which is the quotient of the trivial holomorphic vector bundle $S^2 \times U^{\mathbb{C}}$ through $(\{J\} \times \rho(\ker(J + i)))_{J \in S^2}$; furthermore, \mathcal{U} determines (U, F, ρ) [8, 9].

Returning to the general case, let Z be the twistor space of the quaternionic object (M, F, ρ, c) . Then, for any $x \in M$, the normal bundle of the corresponding twistor sphere $\psi(Y_x) \subseteq Z$, where $\psi : Y \rightarrow Z$ is as above, is (isomorphic to) the holomorphic vector bundle of $(T_x M, E_x, \rho_x)$. Consequently, by applying classical results (see [11] and the references therein) we obtain that $(\psi(Y_x))_{x \in M}$ is contained by a locally complete family of embedded Riemann spheres parameterized by a complexification of M (in particular, M is real-analytic); the members of this family are the *twistor spheres*. Conversely, up to a conjugation and on restricting, if necessary, to an open subset, *any complex manifold endowed with a locally complete family of Riemann spheres, with positive normal bundles, is the twistor space of a quaternionic object* (consequence of [11]).

There is only one more ingredient needed to start describing the Penrose transform for quaternionic objects: a holomorphic line bundle over the twistor space.

Let Z be the twistor space of the quaternionic object (M, F, ρ, c) . Denoting by K_Z the canonical line bundle of Z (that is, K_Z is the determinant of the holomorphic cotangent bundle of Z) and with $\psi : Y \rightarrow Z$ as above, we may assume, by passing, if necessary, to an open neighbourhood of any point of M , that there exists a line bundle \mathcal{L} over Z which is a root of K_Z and whose restriction to each twistor sphere has Chern number -1 . Furthermore, as the antipodal map on Y induces an involutive antiholomorphic involution $\sigma : Z \rightarrow Z$, without fixed points, we have that σ is covered by an antiholomorphic involution $\tau_Z : K_Z \rightarrow K_Z$. It follows that, we may assume that there exists an antiholomorphic anti-involution $\tau : \mathcal{L} \rightarrow \mathcal{L}$ covering σ . Note that, if we restrict \mathcal{L} to the real twistor spheres, then τ give antiholomorphic anti-involutions as in Section 1. Also, for classical quaternionic manifolds of dimensions at least eight it is known that \mathcal{L}^2 exists globally [16], [12].

Remark 2.4. *The quaternionic objects are abundant.* Various classes of examples can be found in [10], [11]. Furthermore, if the twistor space of a quaternionic object is a complex projective manifold then it is a ‘rationally connected manifold’, one of the main objects of study of algebraic geometry (see [7], and the references therein).

3. QUATERNIONIC PLURIHARMONICITY

Let (M, F, ρ, c) be a quaternionic object with twistor space Z . Suppose that \mathcal{L} is a holomorphic line bundle over Z whose restriction to each twistor sphere has Chern number -1 , and which is endowed with an anti-holomorphic anti-involution τ which covers the conjugation σ on Z induced by the antipodal map on Y . As already mentioned, such a line bundle always exists locally (that is, if we pass to an open subset of M). Let H be the dual of the direct image through π of $\psi^*(\mathcal{L}^*)$; that is, the space of sections of H^* over each open set $U \subseteq M$ is the space of sections of $\psi^*(\mathcal{L}^*)$, over $\pi^{-1}(U)$, which are holomorphic when restricted to the fibres of π .

We may assume $F^{\mathbb{C}} = H \otimes E$, where E is a complex vector bundle. Indeed, the

tensor product of $\psi^*(\mathcal{L}^*)$ and \mathcal{B} (note that, the latter is equal to the quotient of $(\ker d\psi)^\mathbb{C}$ through $(\ker d\pi)^{0,1}$) is trivial when restricted to each fibre of π . Therefore $\psi^*(\mathcal{L}^*) \otimes \mathcal{B} = \pi^*(E^*)$, for some vector bundle E over M . Now (cf. [11]), $F^\mathbb{C}$ is equal to the dual of the direct image through π of $\mathcal{B}^* = \psi^*(\mathcal{L}^*) \otimes \pi^*(E)$ and is therefore equal to $H \otimes E$.

Note that (compare Remark 1.2), $Y = PH$, and $\psi^*(\mathcal{L} \setminus 0) = H \setminus 0$, as principal bundles over Y , with structural group \mathbb{C}^* . We shall denote by $\psi_H : H \setminus 0 \rightarrow \mathcal{L} \setminus 0$ the corresponding bundle-map.

Furthermore, $H \setminus 0$ is a principal bundle over M , with group \mathbb{H}^* ; that is, H is a hypercomplex vector bundle of real rank four over M . Indeed, for any $u \in H \setminus 0$ and $z_1 + jz_2 \in \mathbb{H}^*$, we may define $u \cdot (z_1 + jz_2) = z_1 u + z_2 \tau u$.

Let $\pi_H : H \rightarrow M$ and $p : H \setminus 0 \rightarrow Y$ be the projections. We have the following diagram:

$$(3.1) \quad \begin{array}{ccccc} & & H \setminus 0 & & \\ & \swarrow \psi_H & \downarrow p & \searrow \pi_H & \\ \mathcal{L} \setminus 0 & & Y (= PH) & & M \\ \downarrow & \swarrow \psi & \searrow \pi & & \\ Z & & & & \end{array}$$

Obviously, $\mathcal{C}_H = (d\psi_H)^{-1}(T^{0,1}(\mathcal{L} \setminus 0))$ is a co-CR structure on $H \setminus 0$. (Note that, $dp(\mathcal{C}_H)$ is the co-CR structure involved in Definition 2.2.) If $q \in \mathbb{H}^*$ we denote by $[q]$ its image under the projection onto $\mathbb{H}^*/\mathbb{C}^* = \mathbb{CP}^1$ (where $\mathbb{C}^* \subseteq \mathbb{H}^*$ is acting to the right on \mathbb{H}^*). Then $[q] \mapsto \mathcal{C}_H \cdot q^{-1}$ is a family of co-CR structures on $H \setminus 0$ parametrized by the Riemann sphere. It follows that $H \setminus 0$ is a quaternionic object such that its twistor space $Z(H \setminus 0)$, with respect to its underlying smooth structure, is diffeomorphic to $S^2 \times (\mathcal{L} \setminus 0)$. Furthermore, \mathbb{H}^* acts by twistorial diffeomorphisms on $H \setminus 0$ (corresponding to holomorphic diffeomorphisms on $Z(H \setminus 0)$).

In fact, $H \setminus 0$ is a special type of quaternionic object which we define, next.

Definition 3.1. A quaternionic object (M, F, ρ, c) is *hypercomplex* if the bundle Y of admissible linear complex structures on F is trivial and c is the corresponding trivial flat connection.

For the twistor space Z of a hypercomplex object (M, F, ρ, c) , defined by $\psi : Y \rightarrow Z$, we shall assume that the projection $Y = M \times S^2 \rightarrow S^2$ factorises into ψ followed by a holomorphic submersion from Z onto S^2 (obviously, this is automatically satisfied if the fibres of ψ are connected).

Note that, a hypercomplex object (M, F, ρ, c) with ρ an isomorphism is a classical hypercomplex manifold.

Proposition 3.2. *Let Z be the twistor space of a quaternionic object (M, F, ρ, c) defined by $\psi : Y \rightarrow Z$. If the fibres of ψ are connected then the following assertions are equivalent:*

- (i) *There exists a ρ -connection c_1 with respect to which (M, F, ρ, c_1) is hypercomplex.*
- (ii) *There exists a holomorphic submersion $\varphi : Z \rightarrow S^2$ whose restriction to each twistor sphere is a diffeomorphism and which intertwines the conjugation on Z and the antipodal map on S^2 .*

Proof. If (ii) holds, then $\varphi \circ \psi : Y \rightarrow S^2$ induces a trivialization $M = Y \times S^2$. Conversely, if (i) holds then the fibres of ψ are contained in the fibres of the projection $Y \rightarrow S^2$ induced by the trivialization $Y = M \times S^2$. Thus, $Y \rightarrow S^2$ factorises as $\varphi \circ \psi$, where φ is as in (ii). \square

For a hypercomplex object (M, F, ρ, c) with twistor space Z we have a distinguished class of holomorphic line bundles over Z , namely those which are pull backs through φ of holomorphic line bundles over S^2 . Any such holomorphic line bundle \mathcal{L} is characterised by the fact that $\psi^*\mathcal{L}$ is trivial along the fibres of the projection $\varphi \circ \psi : Y \rightarrow S^2$.

Proposition 3.3. *Let Z be the twistor space of a hypercomplex object (M, F, ρ, c) , with a decomposition $F^\mathbb{C} = H \otimes E$ corresponding to a holomorphic line bundle \mathcal{L} over Z .*

Then there exists a unique ρ -connection on H which induces the trivial flat connection on Y and which induces an ‘anti-self-dual’ connection on $\Lambda^2 H$ (that is, flat along the admissible co-CR structures defined by the constant sections of $Y (= M \times S^2)$).

Proof. On identifying $S^2 = \mathbb{CP}^1$, assume, firstly, that $\mathcal{L} = \varphi^*(\mathcal{O}(-1))$, and recall that the space of sections of H^* over any open subset $U \subseteq M$ is, by definition, the space of sections of \mathcal{L}^* over $\pi^{-1}(U)$ which are holomorphic when restricted to the fibres of π . We shall denote in the same way the local sections of H^* and the corresponding local sections of \mathcal{L}^* .

Let $X \in T_x M$, ($x \in M$), and let \tilde{X} be tangent to the fibres of $\varphi \circ \psi$, along $\pi^{-1}(x)$, and such that $d\pi(\tilde{X}) = X$. As $\psi^*\mathcal{L}$ is trivial along the fibres of $\varphi \circ \psi$ we may define $\tilde{X}(s)$, for any local section s of H^* , thus obtaining a holomorphic section of \mathcal{L}^* along $\pi^{-1}(x)$; that is, an element of H_x^* . We have, thus, obtained a connection on H^* with respect to which the sections obtained as pull backs through $\varphi \circ \psi$ of holomorphic sections of $\mathcal{O}(1)$ are covariantly constant. As these sections generate each fibre of H^* , this shows that the obtained connection on H^* is trivial. Dualizing, we obtain a trivializing flat connection on H .

Now, let \mathcal{L}_1 be any other holomorphic line bundle over Z , whose restriction to each twistor sphere has Chern number -1 , and endowed with an antiholomorphic anti-involution covering σ . Let $F^\mathbb{C} = H_1 \otimes E_1$ be the induced decomposition. Then the Ward transform admits a straight generalization to this setting to show that $\mathcal{L}_1 \otimes \mathcal{L}^*$

corresponds to a complex line bundle L , over M , endowed with an anti-self-dual ρ -connection. Consequently, $H_1 = H \otimes L$, which on endowing with the tensor product ρ -connection completes the proof. \square

Remark 3.4. Let Z be the twistor space of a quaternionic object (M, F, ρ, c) , with a decomposition $F^{\mathbb{C}} = H \otimes E$ corresponding to a holomorphic line bundle \mathcal{L} over Z .

As we have seen, $H \setminus 0$ is a hypercomplex object so that there exists a holomorphic submersion φ from its twistor space $Z(H \setminus 0)$ onto $\mathbb{C}P^1$.

Also, $\pi_H : H \setminus 0 \rightarrow M$ is a twistorial map, corresponding to a surjective holomorphic submersion from $Z(H \setminus 0)$ onto Z . Then the pull back through this submersion of \mathcal{L} is $\varphi^*(\mathcal{O}(-1))$.

By using Proposition 3.3 and Remark 3.4, we can now define the relevant ‘quaternionic pluriharmonic’ sheaf, in two steps, as follows.

Definition 3.5. Let (M, F, ρ, c) be a hypercomplex object with twistor space Z , and a decomposition $F^{\mathbb{C}} = H \otimes E$ corresponding to a holomorphic line bundle \mathcal{L} over Z .

Let I and J be anti-commuting admissible linear complex structures on F given by constant sections of $Y(= PH = M \times S^2)$. Then $\mathcal{C} = \rho(\ker(I + i))$ is a complex distribution on M over which the ρ -connection on H induces a partial connection. As \mathcal{C} is integrable, we may define the exterior covariant differential $\bar{\partial}$ mapping $(\Lambda^2 H)$ -valued r -forms on \mathcal{C} to $(\Lambda^2 H)$ -valued $(r + 1)$ -forms on \mathcal{C} , $(r \in \mathbb{N})$.

A section f of $\Lambda^2 H$ is *hypercomplex pluriharmonic* (with respect to the pair (I, J)) if $(\bar{\partial} \circ J \circ \partial)(f) = 0$.

For the particular case of (classical) hypercomplex manifolds, the pluriharmonic functions, also, appear in [2]. However, the operator defining them is due to [20].

Remark 3.6. With the same notations as in Definition 3.5, suppose that the connection on H together with some ρ -connection on E induces a torsion free ρ -connection ∇ on F . Then a quick calculation shows that, for any section f of $\Lambda^2 H$, the following assertions are equivalent (recall that, in dimension four, a linear quaternionic structure is just an oriented linear conformal structure):

- (i) f is hypercomplex pluriharmonic;
- (ii) $\nabla^2 f$ restricted to any quaternionic subspace, of real dimension four, of any fibre of F , is trace free.

In particular, in this case, the notion of hypercomplex pluriharmonicity does not depend of the pair (I, J) .

Definition 3.7. Let (M, F, ρ, c) be a quaternionic object with twistor space Z , and a decomposition $F^{\mathbb{C}} = H \otimes E$ corresponding to a holomorphic line bundle \mathcal{L} over Z .

A section f of $\Lambda^2 H$ is *quaternionic pluriharmonic* if the corresponding equivariant function $\tilde{f} : H \setminus 0 \rightarrow \mathbb{C}$ is hypercomplex pluriharmonic.

In the particular cases of hyper-Kähler and quaternionic manifolds, our definition reduces to the ones considered in [5] and [1], respectively (note that, the latter is based on [3]).

The fact that \mathbb{H}^* acts by twistorial diffeomorphisms on $H \setminus 0$ implies that Definition 3.7 does not depend of the choice of I and J on $H \setminus 0$.

4. THE PENROSE TRANSFORM

Let (M, F, ρ, c) be a quaternionic object with twistor space Z , and a decomposition $F^{\mathbb{C}} = H \otimes E$ corresponding to a holomorphic line bundle \mathcal{L} over Z . We make the following definition, where $\psi : Y(= PH) \rightarrow Z$ is the corresponding surjective co-CR submersion (recall the diagram (3.1)).

Definition 4.1. The *Penrose transform* is the complex linear map associating to each $\gamma \in H^1(Z, \mathcal{L}^2)$ the section $(\int_{\pi^{-1}(x)} \psi^* \gamma)_{x \in M}$ of $\Lambda^2 H$.

The injectivity of the Penrose transform is quite straightforward.

Proposition 4.2. *If the fibres of ψ are connected then, by passing to an open neighbourhood of each point of M , the Penrose transform is injective.*

Proof. If M is a hypercomplex object then this follows quickly from standard arguments, involving Čech cohomology and by using [17]. In general, through the pull back, the proof for $H \setminus 0$ gives, in particular, the required injectivity, locally, on M . \square

We say that the quaternionic object (M, F, ρ, c) is of *constant type* if the co-CR quaternionic vector spaces $(T_x M, F_x, \rho_x)$, $(x \in M)$, are the same, up to isomorphisms.

Here is the main result of this paper.

Theorem 4.3. *Let (M, F, ρ, c) be a quaternionic object with twistor space Z , and a decomposition $F^{\mathbb{C}} = H \otimes E$ corresponding to a holomorphic line bundle \mathcal{L} over Z .*

The following statements hold:

(i) *If on $H \setminus 0$ the sheaf of quaternionic pluriharmonic functions is equal to the sheaf of hypercomplex pluriharmonic functions (in particular, if $H \setminus 0$ admits a compatible torsion free ρ -connection) then the image of the Penrose transform is contained in the space of quaternionic pluriharmonic sections of $\Lambda^2 H$.*

(ii) *If (M, F, ρ, c) is of constant type, then, by passing, if necessary, to an open neighbourhood of each point of M , the Penrose transform admits a partial section defined on the space of quaternionic pluriharmonic sections of $\Lambda^2 H$.*

The proof of Theorem 4.3, essentially, relies on the fact that it is sufficient to prove it for hypercomplex objects. To show this, we need the following result.

Proposition 4.4. *Let (M, F, ρ, c) be a quaternionic object with twistor space Z , and a decomposition $F^{\mathbb{C}} = H \otimes E$ corresponding to a holomorphic line bundle \mathcal{L} over Z .*

Then there exists a principal ρ -connection $C : \pi_H^ F \rightarrow T(H \setminus 0)$ on $(H \setminus 0, M, \mathbb{H}^*)$*

such that the associated ρ -connection on $Y(=PH)$ induces the same co-CR structure on Y as the one defining $\psi : Y \rightarrow Z$.

Proof. This is a consequence of the fact that $\pi_H : H \setminus 0 \rightarrow M$ is twistorial and in particular, at each point, its differential is a co-CR quaternionic linear map (see [8, 9] for the definition of the latter). Therefore, on denoting by (F_H, ρ_H) the almost co-CR quaternionic structure of $H \setminus 0$ we have $d\pi_H \circ \rho_H = \rho \circ \Pi$ where $\Pi : F_H \rightarrow F$ is a morphism of quaternionic vector bundles, covering π_H . Furthermore, Π factorises into a morphism of quaternionic vector bundles $\tilde{\Pi} : F_H \rightarrow (\pi_H)^*(F)$ followed by the bundle-map $(\pi_H)^*(F) \rightarrow F$.

Now, $(\pi_H)^*(F^{\mathbb{C}}) = ((H \setminus 0) \times \mathbb{C}^2) \otimes E$, whilst $F_H^{\mathbb{C}} = ((H \setminus 0) \times \mathbb{C}^2) \otimes E_H$, for some complex vector bundle E_H over $H \setminus 0$, such that, $\tilde{\Pi}$ is given by the tensor product of the identity morphism of $(H \setminus 0) \times \mathbb{C}^2$ and a morphism of complex vector bundles from E_H onto $(\pi_H)^*(E)$. Hence, by taking a section of the latter, we may embed $(\pi_H)^*(F) \subseteq F_H$, as a quaternionic subbundle, so that the restriction of $\tilde{\Pi}$ to it is the identity morphism.

On the other hand, as a structural group, \mathbb{H}^* induces right actions on F_H and on $T(H \setminus 0)$, where the former is given by the action on (the first components) of $(\pi_H)^*(H) = (H \setminus 0) \times \mathbb{C}^2$. Moreover, ρ_H is equivariant with respect to these actions.

Then the restriction C of ρ_H to $(\pi_H)^*(F)$ is a principal ρ -connection, as claimed. \square

Another ingredient needed to prove Theorem 4.3 is a natural extension, to this setting, of the classical notion of ‘standard (horizontal) vector field’. For this, let (M, F, ρ, c) be a quaternionic object with twistor space Z , and a decomposition $F^{\mathbb{C}} = H \otimes E$ corresponding to a holomorphic line bundle \mathcal{L} over Z . Endow $H \setminus 0$ with a principal ρ -connection C as in Proposition 4.4.

Definition 4.5. For any $\xi \in \mathbb{C}^2$ let $B(\xi)$ be the morphism of complex vector bundles, from $(\pi_H)^*(E)$ to $T^{\mathbb{C}}(H \setminus 0)$ given by $B(\xi)(u, s) = C(u, (u\xi) \otimes s)$, for any $u \in H \setminus 0$ and $s \in E_{\pi_H(u)}$.

Proposition 4.6. On denoting by R_q the right translation by $q \in \mathbb{H}^*$ on $H \setminus 0$ (and $(\pi_H)^*(E)$), we have $dR_q \circ B(\xi) = B(q^{-1}\xi) \circ R_q$, for any $\xi \in \mathbb{C}^2$.

Proof. Let $u \in H \setminus 0$ and $s \in E_{\pi_H(u)}$. By the equivariance of C , we have

$$dR_q(B(\xi)(u, s)) = dR_q(C(u, (u\xi) \otimes s)) = C(u \cdot q, (u\xi) \otimes s).$$

On the other hand,

$$(B(q^{-1}\xi) \circ R_q)(u, s) = B(q^{-1}\xi)(u \cdot q, s) = C(u \cdot q, ((u \cdot q)(q^{-1}\xi)) \otimes s) = C(u \cdot q, (u\xi) \otimes s),$$

thus completing the proof. \square

The morphism of Lie algebras from \mathbb{H} (seen as the Lie algebra of \mathbb{H}^*) to the Lie algebra of vector fields on $T(H \setminus 0)$ extends to a morphism of complex Lie algebras

from $A \in \mathfrak{gl}(2, \mathbb{C})$ to the sections of $T^{\mathbb{C}}(H \setminus 0)$. We shall denote in the same way, the elements of $\mathfrak{gl}(2, \mathbb{C})$ and the corresponding (fundamental) complex vector fields on $H \setminus 0$. Then, from Proposition 4.6, we quickly obtain the following fact.

Corollary 4.7. *If $\xi \in \mathbb{C}^2$ and s is a section of E then the vector field $B(\xi)(s)$, which at any $u \in H \setminus 0$ is equal to $B(\xi)(u, s_{\pi_H(u)})$, satisfies $[A, B(\xi)(s)] = B(A\xi)(s)$, for any $A \in \mathfrak{gl}(2, \mathbb{C})$.*

We now have a convenient way to write $\mathcal{C}_H = (\mathrm{d}\psi_H)^{-1}(T^{0,1}(\mathcal{L} \setminus 0))$, namely, $\mathcal{C}_H = \mathcal{B}_H \oplus (\ker \mathrm{d}\pi_H)^{0,1}$, where \mathcal{B}_H is generated by $B(e_1)(s)$, for any (local) section s of E , with (e_1, e_2) the canonical basis of \mathbb{C}^2 .

Proposition 4.8. *Let (M, F, ρ, c) be a quaternionic object with twistor space Z , and a decomposition $F^{\mathbb{C}} = H \otimes E$ corresponding to a holomorphic line bundle \mathcal{L} over Z . Endow $H \setminus 0$ with a principal ρ -connection C as in Proposition 4.4.*

Then a section f of $\Lambda^2 H$ is quaternionic pluriharmonic if and only if the corresponding equivariant function \tilde{f} on $H \setminus 0$ satisfies $((\bar{\partial} \circ J \circ \partial)(\tilde{f}))(X, Y) = 0$, for any $X, Y \in \mathcal{B}_H$.

Proof. The necessity of that condition for f to be quaternionic pluriharmonic is trivial. For the sufficiency, firstly, note that f is quaternionic pluriharmonic if and only if

$$(4.1) \quad X((JY)(\tilde{f})) - Y((JX)(\tilde{f})) - (J[X, Y])(\tilde{f}) = 0,$$

for any sections X and Y of \mathcal{C}_H .

By writing \tilde{f} with respect to a local section of $H \setminus 0$, a straightforward calculation shows that (4.1) always holds for any X and Y sections of $(\ker \mathrm{d}\pi_H)^{0,1}$.

Note that, $(\ker \mathrm{d}\pi_H)^{0,1}$ is generated by (the fundamental complex vector fields corresponding to) those $A \in \mathfrak{gl}(2, \mathbb{C})$ for which the first column is zero. Take X to be such an A , and $Y = B(e_1)(s)$ for some section s of E . Then Corollary 4.7 shows that $[X, Y] = 0$. Further, JX is given by the matrix whose first column is the opposite of the second column of A , whilst its second column is zero. Also, $JY = B(e_2)(s)$.

On denoting by $\xi (\in \mathbb{C}^2)$ the second column of A we have that $(JX)(\tilde{f}) = \xi_1 \tilde{f}$. Hence, $Y((JX)(\tilde{f})) = (B(\xi_1 e_1)(s))(\tilde{f})$.

Now, by applying, again, Corollary 4.7, we obtain that $[X, JY] = B(\xi)(s)$. Hence,

$$\begin{aligned} X((JY)(\tilde{f})) &= JY((X)(\tilde{f})) + (B(\xi)(s))(\tilde{f}) \\ &= -\xi_2 (B(e_2)(s))(\tilde{f}) + (B(\xi)(s))(\tilde{f}) = (B(\xi_1 e_1)(s))(\tilde{f}). \end{aligned}$$

We have, thus, shown that (4.1) is automatically satisfied, also, when X is a section $(\ker \mathrm{d}\pi_H)^{0,1}$ and Y is a section of \mathcal{B}_H . The proof is complete. \square

Remark 4.9. A quick consequence of Proposition 4.8 is that, for a hypercomplex object, the quaternionic pluriharmonic sheaf is the intersection of the hypercomplex pluriharmonic sheaves, with respect to all possible (anti-commuting) pairs (I, J) .

Consequently, the following assertions are equivalent, for a hypercomplex object:

- (i) The hypercomplex pluriharmonicity does not depend of the pair (I, J) .
- (ii) The hypercomplex and the quaternionic pluriharmonicity coincide.

Next, we give the following:

Proof of Theorem 4.3. To prove (i), by passing, through pull back, to $H \setminus 0$, we may suppose M hypercomplex and, on it, the sheaf of quaternionic pluriharmonic sections of $\Lambda^2 H$ is equal to the sheaf of hypercomplex pluriharmonic sections of $\Lambda^2 H$. Then, by using Čech cohomology, integral formulae similar to (0.1) can be, locally, obtained for the sections of $\Lambda^2 H$ which are in the image of the Penrose transform. Consequently, the image of the Penrose transform is contained in the space of quaternionic pluriharmonic sections of $\Lambda^2 H$.

To prove (ii), we may complexify M so that H and (consequently) Y are the restrictions to M of holomorphic bundles over $M^{\mathbb{C}}$. For simplicity, we denote with the same H and Y , respectively, these bundles over $M^{\mathbb{C}}$; also, the same for the notation of the corresponding bundle projections. We, thus, have the same diagram as (3.1), up to the fact that M is replaced by $M^{\mathbb{C}}$. Furthermore, we have the following diagram (which, in part, is the complexification of (3.1)),

$$(4.2) \quad \begin{array}{ccccc} & & \text{GL}(H) & & \\ & \swarrow \psi_{H,j} & \downarrow p & \searrow \pi_H & \\ \mathcal{L} \setminus 0 & & (Y + Y) \setminus Y & & M^{\mathbb{C}} \\ & \searrow \psi_j & \downarrow \pi & \swarrow \pi & \\ & Z & & & \end{array}$$

where $\text{GL}(H)$ is the frame bundle of H , $Y (= PH)$ is diagonally embedded into $Y + Y$, whilst $\psi_{H,j}$ is the composition of ψ_H with the projection onto the j -component of any frame on H , and similarly for ψ_j , ($j = 1, 2$), and we have, also, denoted by π_H the projection from $\text{GL}(H)$ onto $M^{\mathbb{C}}$.

Let ∂_j be the partial exterior differential determined by the holomorphic foliation \mathcal{C}_j given by (the connected components of) the fibres of $\psi_{H,j}$, ($j = 1, 2$). As $\text{GL}(H)$ is the complexification of $(H \setminus 0)|_M$, we obtain (through a complexification) a morphism of holomorphic vector bundles $J : \mathcal{C}_2^* \rightarrow \mathcal{C}_1^*$. Then a holomorphic section f of $\Lambda^2 H$ is (complex-)quaternionic pluriharmonic if and only if the corresponding equivariant holomorphic function \tilde{f} on $\text{GL}(H)$ satisfies $(\partial_1 \circ J \circ \partial_2)(\tilde{f}) = 0$.

Note that, $\text{GL}(H)$, as a bundle over $(Y + Y) \setminus Y$, is a principal bundle with group $\mathbb{C}^* \times \mathbb{C}^*$, where the action is induced, by restriction to the diagonal matrices, from the action of $\text{GL}(2, \mathbb{C})$. Furthermore, $\psi_{H,j}$ is a morphism of principal bundles, covering ψ_j , with respect to the morphism of Lie groups $\mathbb{C}^* \times \mathbb{C}^* \rightarrow \mathbb{C}^*$, $(\lambda_1, \lambda_2) \mapsto \lambda_j$, ($j = 1, 2$).

Let f be a section of $\Lambda^2 H$ and let \tilde{f} be the corresponding equivariant function on

$\mathrm{GL}(H)$. Then, for any $(\lambda_1, \lambda_2) \in \mathbb{C}^* \times \mathbb{C}^*$, we have

$$(4.3) \quad R_{(\lambda_1, \lambda_2)}^*((J \circ \partial_2)(\tilde{f})) = \lambda_1^{-2}(J \circ \partial_2)(\tilde{f}),$$

where R^* denotes the pull back transformation induced by the right translation. This is straightforward if we restrict to the fibres of π_H . Further, the proof of Proposition 4.4 works in this setting as well by passing, if necessary, to an open neighbourhood of each point of M . Consequently, Proposition 4.6 can be easily adapted to this setting, and by using it (4.3) quickly follows.

Consequently, if f is a quaternionic pluriharmonic section of $\Lambda^2 H$ then the partial 1-form $(J \circ \partial_2)(\tilde{f})$ defines a 1-cocycle, for the relative de Rham cohomology (see [4]) of \mathcal{L}^2 , with respect to ψ_1 , whose first group we denote by $H_{\psi_1}^1(Z, \mathcal{L}^2)$. Consequently, we have obtained an injective linear map, into this cohomology group, from the space of quaternionic pluriharmonic sections of $\Lambda^2 H$ (on integrating $(J \circ \partial_2)(\tilde{f})$ along the fibres of π_H we obtain f , up to a nonzero constant factor).

Therefore (ii) is proved if, locally, we have $H^1(Z, \mathcal{L}^2) = H_{\psi_1}^1(Z, \mathcal{L}^2)$. From the main result of [4], we deduce that, locally, this holds if the fibres of ψ are contractible. Now, if (M, F, ρ, c) is of constant type, we may assume that the images through π_H of the fibres of ψ are totally geodesic with respect to a (classical) connection on $M^{\mathbb{C}}$. Thus, in this case, by passing to any convex neighbourhood over which Y is trivial, we have the desired cohomology isomorphism, and the proof is complete. \square

Remark 4.10. From Proposition 4.2 and Theorem 4.3 we obtain that, locally, the Penrose transform is an isomorphism in the following cases:

- (1) for (classical) quaternionic manifolds [3] (see, also, [6], for the particular case of anti-self-dual manifolds).
- (2) for co-CR quaternionic vector spaces (this was, essentially, proved in [18]).

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E-mail address: `radu.pantilie@imar.ro`

R. PANTILIE, INSTITUTUL DE MATEMATICĂ “SIMION STOILOW” AL ACADEMIEI ROMÂNE, C.P. 1-764, 014700, BUCUREȘTI, ROMÂNIA